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Interior point solutions of variational problems and global inverse function theorems

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SUMMARY

Variational problems and the solvability of certain nonlinear equations have a long and rich history beginning with calculus and extending through the calculus of variations. In this paper, we are interested in 'well-connected' pairs of such problems which are not necessarily related by critical point considerations. We also study constrained problems of the kind which arise in mathematical programming. We are also interested in interior minimizing points for the variational problem and in the well-posedness (in the sense of Hadamard) of solvability of the related systems of equations. We first prove a general result which implies the existence of interior points and which also leads to the development of certain generalization of the Hadamard-type global inverse function theorem, along the theme that uniqueness quite often implies existence. This result is illustrated by proving the non-existence of shock waves for certain initial data for the vector Burgers' equation. The global inverse function theorem is also illustrated by a derivation of the existence of positive definite solutions of matrix Riccati equations without first analysing the nonlinear matrix Riccati differential equation. The main results on the existence of solutions to geometrically constrained well-connected pairs are then presented and illustrated by a geometric analysis of the existence of interior points for linear programming problems. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Vladimir Andreevich Yakubovich is one of the founding fathers of modern control theory with pioneering and fundamental contributions in stability theory and optimal control, too many to

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seriously recount here. This paper is written in homage to our esteemed colleague and friend Vladimir Andreevich on the occasion of his 80th birthday, and, at least indirectly, it connects up to some of his work on nonlinear analysis [1, 2], linear-quadratic optimization problems [3, 4] and output feedback by interpolation [5, 6].

Consider two n-manifolds M and N and a continuous map

$$f : M \to N \tag{1}$$

We assume that N is connected. Following Hadamard, one is typically interested in the existence of solutions to the equations

$$f(x) = y \tag{2}$$

the uniqueness of solutions and the continuous dependence of solutions on the y. More formally, the problem of finding solutions to (2) is said to be *well-posed* provided the map (1) satisfies the following:

(i) f is surjective.
(ii) f is injective.
(iii) f has a continuous inverse.

In geometric terms, the solvability problem is well-posed provided f is a homeomorphism. One consequence of well-posedness is therefore that the map f is *proper*; i.e. that, for any compact subset K of $N, f^{-1}(K)$ is compact. In other words, given *a priori* bounds on the right-hand side of (2), there exists bounds on the solutions to (2).

We shall also consider variational problems of the following form. For each $y \in N$, let $\mathcal{Q}_y$ be a submanifold (possibly with boundary) of a smooth manifold $\mathcal{X}_y$. In many applications, $\mathcal{Q}_y$ and $\mathcal{X}_y$ will not vary with y, in which case we suppress the subscripts. For each $y \in N$, suppose there is a function

$$J_y : \mathcal{Q}_y \to \mathbb{R} \cup \{\infty\} \tag{3}$$

Here, one is often interested in computing

$$\inf_{q \in \mathcal{Q}_y} J_y(q) \tag{4}$$

and in finding all q_0 for which

$$J_y(q_0) = \inf_{q \in \mathcal{Q}_y} J_y(q) \tag{5}$$

Denoting by $\text{int}(\mathcal{Q}_y)$ the set of interior points of $\mathcal{Q}_y$ in its topology, we are particularly interested in finding minimizers in $\text{int}(\mathcal{Q}_y)$. We refer to such a point as an *interior minimizing point for* J_y. While for concave problems we are interested in interior maximizing points, there are important problems when we would want to know about the uniqueness of saddle points. For these reasons, and others illustrated throughout this paper, our basic definition will include general critical points.

Definition 1.1
Consider a pair of (families of) problems of the form formulated above. This pair is said to *well-connected* if, for all $y \in N$, there exists an injection from the solution set of (2) into the set of interior critical points of J_y.

Of course, one instance of well-connectedness arises from an analysis of the stationarity of interior points for variational problems. Conversely, there is a finite-dimensional analogue of

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the inverse problem, or Dirichlet Principle, of the calculus of variations. More explicitly, suppose

$$f:\mathbb{R}^n \to (\mathbb{R}^n)^*$$

is a smooth map. This map defines a 1-form on $\mathbb{R}^n$ which can be expressed as

$$\omega = \sum_{i=1}^{n} f_i \, \mathrm{d}x_i$$

in terms of a global co-ordinate system $(x_1, x_2, \ldots, x_n)$. If $\mathrm{d}\omega = 0$, ω is closed on $\mathbb{R}^n$ and hence exact; i.e.

$$\mathrm{d}J = \omega$$

for some smooth $J:\mathbb{R}^n \to \mathbb{R}$. Since

$$D^2J(x) = \mathrm{Jac}(f)(x)$$

$\mathrm{Jac}(f)$ is symmetric, and we now assume that $\mathrm{Jac}(f)(x) > 0$ for all $\in \mathbb{R}^n$ so that

$$J_y(x) = J(x) - y(x)$$

is strictly convex for each $y \in (\mathbb{R}^n)^*$. Alternatively, by the second derivative test, J_y has only minima as critical points, and to say that J_y achieves its minimum at x is to say that $f(x) = y$. Now suppose f is proper. By Hadamard's global inverse function theorem [7–9], $f:\mathbb{R}^n \to (\mathbb{R}^n)^*$ is a diffeomorphism so that the problem, $f(x) = y$, is well-posed, and, in fact, J_y always achieves its minimum.

As an elementary example, consider the equation

$$F(z) = c \tag{6}$$

where F is a holomorphic function of one variable and $c = c_1 + \mathrm{i}c_2$. Since the 1-form $(F(z) - c)\,\mathrm{d}z$ is exact, taking the imaginary part of a primitive yields a functional $J_c:\mathbb{C} \to \mathbb{R}$ such that

$$\mathrm{d}J_c = (v - c_2)\,\mathrm{d}x + (u - c_1)\,\mathrm{d}y$$

where $F(x, y) = u(x, y) + \mathrm{i}v(x, y)$. In particular, the pair (F, J_c) is well-connected. Moreover, by the Cauchy–Riemann equations, the Hessian D^2J_c is an indefinite symmetric matrix satisfying

$$\det D^2J_c \leqslant 0$$

with equality at z_0 if and only if $F'(z_0) = 0$. In particular, if $F'(z) \neq 0$, (6) is well-connected to a variational problem for which the critical points are non-degenerate saddle points. Interestingly, J_c has a unique critical point for all c if and only if F is univalent.

Remarkably, a similar situation arises in the analysis of the generalized moment problem of classical analysis. For simplicity, we restrict to the case of real quantities. Consider a sequence of real numbers $c_0, c_1, \ldots, c_n$ and a sequence of continuous, linearly independent real-valued functions $\alpha_0, \alpha_1, \ldots, \alpha_n$ defined on the real interval $[a, b]$. The moment problem is then to find all monotone, non-decreasing functions μ of bounded variation such that

$$\int_a^b \alpha_k(t)\,\mathrm{d}\mu(t) = c_k, \quad k = 0, 1, \ldots, n \tag{7}$$

where the sequence $c_0, c_1, \ldots, c_n$ is positive in the following sense. Let $\mathfrak{P}$ be the subspace of $C[a, b]$ spanned by the functions

$$\alpha_0, \alpha_1, \ldots, \alpha_n$$

and let $\mathfrak{P}_+$ be the subset of $P \in \mathfrak{P}$ that are positive on $[a, b]$. It is typically assumed that $\mathfrak{P}_+$ is non-empty, in which case $\mathfrak{P}_+$ is an open convex subspace of $\mathfrak{P}$. One says that the sequence $c_0, c_1, \ldots, c_n$ is *positive* if and only if

$$\langle c, q \rangle := \sum_{k=0}^{n} q_k c_k > 0 \tag{8}$$

for all $q := (q_0, q_1, \ldots, q_n) \in \mathbb{R}^{n+1}$ such that

$$Q := \sum_{k=0}^{n} q_k \alpha_k \in \mathfrak{P}_+ \tag{9}$$

Denote by $\mathfrak{C}_+$ the space of positive sequences. The fact that $\mathfrak{P}_+$ is non-empty implies that $\mathfrak{C}_+$ is non-empty. Therefore, $\mathfrak{C}_+$ is also a convex open subset of the space of real sequences of length $n + 1$.

Motivated by several applications in speech synthesis, robust control and signal processing (see [10–13]), we introduce the complexity constraint

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \Phi(t) = \frac{P(t)}{Q(t)}, \quad P, Q \in \mathfrak{P}_+ \tag{10}$$

Fix P and define the function

$$F : \mathfrak{P}_+ \to \mathfrak{C}_+$$

componentwise via

$$F_k(Q) = \int_a^b \alpha_k(t)\, \mathrm{d}\mu(t)$$

Parameterizing Q via $Q = \sum_{k=0}^{n} q_k \alpha_k$, we construct the 1-form

$$\omega_c = \sum_{k=0}^{n} [c_k - F_k(\mu)]\, \mathrm{d}q_k$$

on $\mathfrak{P}_+$ and observe that ω_c is closed. Therefore, by the Poincaré Lemma, there exists a smooth function J_c such that $\mathrm{d}J_c = \omega_c$, leading to the construction of a well-connected pair for the generalized moment problem with complexity constraints. More explicitly,

$$J_c = \int \omega_c$$

with the integral being independent of the path between two endpoints. Therefore, since

$$\omega_c = \sum_{k=0}^{n} \left[c_k - \int_a^b \alpha_k \frac{P}{Q} \mathrm{d}t \right] \mathrm{d}q_k$$

$$= \sum_{k=0}^{n} c_k\, \mathrm{d}q_k - \int_a^b \frac{P}{Q} \mathrm{d}Q\, \mathrm{d}t$$

computing the path integral

$$\int_{Q_0}^{Q_1} \omega_c = \left[\langle c, q \rangle - \int_a^b P \log Q \, \mathrm{d}t \right]_{Q_0}^{Q_1}$$

we obtain, modulo a constant of integration,

$$J_c(Q) = \langle c, q \rangle - \int_a^b P \log Q \, \mathrm{d}t$$

In general, since J_c is strictly convex, any interior critical point is a non-degenerate minimum, yielding uniqueness of a distribution of constrained complexity solving the moment problem. Consequently, we have a well-connected pair of problems consisting of the moment problems

$$\int_a^b \alpha_k(t) \frac{P(t)}{Q(t)} \mathrm{d}t = c_k, \quad k = 0, 1, \ldots, n; \quad c \in \mathfrak{C}_+ \tag{11}$$

and the optimization problems

$$\min_{Q \in \mathfrak{P}_+} J_c(Q), \quad c \in \mathfrak{C}_+ \tag{12}$$

We shall return to this example in Section 2.

As a final example, these conditions are also always satisfied for the pair of well-connected problems arising from the Ritz approximations of a strictly convex variational problem and, as pointed out by Hilbert, its fully elliptic Euler–Lagrange equation. For example, following [14], let G be a bounded region in $\mathbb{R}^N$ with a piecewise smooth boundary ∂G, and let $\bar{G} = G \cup \partial G$ be the closure of G. Consider the variational problem

$$\inf_{u \in C^1(\bar{G})} J_h(u), \quad u = 0 \quad \text{on } \partial G$$

where

$$J_h(u) = \int_G \left(\frac{1}{2} \sum_{k=1}^N \left(\frac{\partial u}{\partial x_k} \right)^2 + g(u) - uh \right) \mathrm{d}x$$

The corresponding Euler–Lagrange equations are given by

$$\Delta u - g'(u) = -h \quad \text{on } G$$

$$u = 0 \quad \text{on } \partial G$$

and they are equivalent to the variational problem, for suitable g and sufficiently smooth u ($u \in C^2(\bar{G})$). If $V \subset C^2(\bar{G})$ is finite-dimensional subspace, Ritz's method gives two problems, namely the optimization problem

$$\inf_{v \in V} \{ \varphi(v) - \langle b, v \rangle \}$$

where φ is a smooth function and $b \in V$, and the problem to solve the equation

$$\varphi'(v) = b$$

In this case, φ is strictly convex, φ' is strictly monotone and proper (coercive), and

$$D^2 \varphi > 0$$

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In this paper, however, we are also interested in well-connected pairs where f need not generate an exact 1-form and need not have an everywhere non-singular Jacobian. In Section 2 we prove a 'duality' theorem for well-connected pairs which asserts that problem (2) is well-posed and the variational problem (4) has an interior critical point provided f is proper and J_y has at most one critical point. Of course, the case when $\mathcal{Q}_y$ is convex and J_y is strictly convex is particularly easy to apply. The underlying construction leads to a new class of global inverse function theorems, which we describe in Section 2.

In Section 3 the duality theorem is illustrated by a well-connected pair of problems concerning shock waves in an interval $[t_0, T]$ for solutions of the (vector) inviscid Burgers' equation. We begin by defining a variational problem for which, when viewed as an optimal control problem, the Riccati partial differential equation [15] for the gradient of the value function turns out to be the Burgers' equation in backwards time. We then construct, for each $t \in [t_0, T]$, a map f_t for which well-posedness is equivalent to the absence of shock waves at time t. Several criteria for properness of this map are given in terms of qualitative behaviour of the initial data. After rendering the variational problem finite dimensional, it is shown that these two problems are well-connected. In particular, convexity of the terminal constraint condition is equivalent to monotonicity of the initial condition, which is related (equivalent in the scalar case) to the absence of shock waves. Moreover, in the non-convex case, the formation of multiple critical points is related to the formation of shock waves. Finally, an example is given where a parameter variation in the initial condition causes an onset of shock waves that is reflected, as a finite-dimensional 'shadow,' in the critical point equation undergoing a pitchfork bifurcation.

The proof of our basic duality theorem in Section 2 belongs to a long tradition of global inverse function theorems, relying on a theorem that uniqueness of solutions to certain finite-dimensional nonlinear problems implies existence. In Section 4 this theorem is illustrated by showing existence of positive definite solutions to the algebraic matrix Riccati equation, for which it is relatively simple to establish uniqueness. In Section 5, we again illustrate the duality theorem, in the context of constrained optimization, for the non-trivial problem of the analysis of interior point methods for finite-dimensional linear programming.

2. GLOBAL INVERSE FUNCTION THEOREMS

Global inverse function theorems have long been implicitly a part of global variational problems. For example, a co-ordinate-free formulation of the minimization problem for a smooth (strictly) convex function can be given as follows. Suppose $U \subset \mathbb{R}^n$ is an open, connected subset and that φ maps U to $\mathbb{R}$ as a smooth, proper function, bounded from below, whose Hessian is everywhere non-singular. Then, the Morse inequalities show that φ always has a unique minimum, while Milnor's Theorem then asserts that U is diffeomorphic to $\mathbb{R}^n$. Indeed, if the derivative of φ is proper, the derivative is one such diffeomorphism, by Hadamard's Theorem (or certain of its generalizations). Of course, if U is convex this is also standard. However, in cases where such information about the Hessian is not available, one could still use global inverse function theorems, of the kind hinted at in the previous section, to deduce the uniqueness and existence of minima and critical points in degenerate cases.

Matters being so, we are interested in a class of global inverse function theorems in which the uniqueness of solutions to (2) can imply existence. That uniqueness implies existence for a class of nonlinear problems has several precedents. Generalizing the linear case, it is known [16, 17] that injective polynomial maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ are surjective. One can also deduce this in the case where f is locally injective and N is simply connected from the Banach–Mazur Theorem [18, p. 221] after one observes, by Brouwer's Theorem, that locally injective maps of n-manifolds are also local homeomorphisms. In the smooth category for the same class of smooth manifolds, Hadamard's Theorem asserts that smooth, proper maps with non-vanishing Jacobian are diffeomorphisms, as we now discuss in more detail.

Theorem 2.1
Suppose M and N are connected, n-dimensional, topological manifolds and that N is simply connected. Consider a continuous map $f : M \to N$. Then, f is a homeomorphism if and only if f is locally injective and proper.

Proof
If f is a homeomorphism, f is of course locally injective and proper. Conversely, by Brouwer's Theorem on Invariance of the Domain [19, p. 3], f is an open mapping locally, and hence globally. Since f is proper, the open submanifold $f(M)$ is also closed in N. Since N is connected, f is surjective as before. Since f is proper, for each $y \in N$, the number of solutions to (2) is finite. Since f is locally injective and proper, f is a local homeomorphism. By a modification of the construction in [20, p. 8] as in [21, p. 758], the number of solutions is independent of y, and $f : M \to N$ is a covering space. Since N is simply connected, f must be injective, and therefore f is a homeomorphism. □

Not surprisingly, a great deal can be said about well-connected pairs of problems, especially when either the number of solutions to (2) or the number of minimizers for J_y is independent of y. One such example is given in the following result.

Theorem 2.2
Consider a well-connected pair of problems. If f is proper, and, for each $y \in N$, J_y has at most one critical point, then

(i) problem (2) is well-posed;
(ii) for each $y \in N$, J_y has a unique critical point which is an interior point.

Proof
Since J_y has at most one critical point, and the pair of problems is well-connected, f is injective. In this case, uniqueness implies the existence of solutions (Theorem 2.1). Matters being so, since the pair of problems is well-connected, for each $y \in N$, J_y has a critical point that is an interior point, which is therefore the unique critical point of J_y. □

To illustrate this theorem, we return to the well-connected pair of problems (11) and (12). Under mild technical conditions, it can be shown that F is proper [22, 23]. Since J_c is strictly convex, it has at most one critical point that must then be a minimizer. Theorem 2.2 then implies that such a unique minimum always exists. Moreover the minimizer is an interior point, and the moment problem (11) is well-posed. A very complete analysis of this moment problem from a

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variational viewpoint is given in [23]. We return below with more general examples of well-connected pairs of problems, where f need not generate an exact 1-form.

Specializing Theorem 2.1 to the case of Euclidean spaces, we obtain an easy corollary.

Corollary 2.3

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map. Then f is a homeomorphism if and only if f is locally injective and proper.

Since local homeomorphisms are locally injective, we obtain a neat result due to Palais [24]; also see [25].

Corollary 2.4 Palais–Hadamard

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map. The f is a homeomorphism if and only if f is a proper, local homeomorphism.

As is well known, this result implies Hadamard's result on the well-posedness of maps between Euclidean spaces.

Corollary 2.5 Hadamard

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^k for $k \geqslant 1$. Then f is a C^k-diffeomorphism if and only if f is proper and $\det\{(\text{Jac} f)(x)\}$ is non-zero for each $x \in \mathbb{R}^n$.

Generalizations of the theorems of Palais and Hadamard to the case of covering spaces are standard; see [18]. The new results we present here have replaced the requirement that the map in question be a local homeomorphism with the simpler hypothesis that the map be locally injective.

Theorem 2.6

Suppose M and N are n-dimensional, topological manifolds and that N is connected. Consider a continuous map $f : M \to N$. Then, f is a homeomorphism if and only if f is injective and proper. In this case, M is connected.

Proof

If f is a homeomorphism, then f is injective, and since f^{-1} is continuous f is proper. Conversely, if f is injective, f is an open mapping by Brouwer's Theorem on Invariance of Domain. In particular, $f(M)$ is a non-empty open submanifold of N. Since f is proper, f is a closed mapping so that $f(M)$ is a closed subset of N. Since N is connected, f is surjective. Finally, since f is a closed mapping, f^{-1} is continuous. □

In Sections 3–5, we illustrate these results in the context of Burgers' equation, the algebraic Riccati equation and interior point methods for linear programming. These examples, as well as the example with the moment problem discussed above suggest that there is a broad class of problems for which the number of interior points for J_y is at most one. We conclude this section with specializations of the duality theorem to the case where the variational problems are convex.

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Corollary 2.7
Consider a well-connected pair of problems for which $\mathcal{Q}_y$ is a convex subset of a topological vector space and for which J_y is strictly convex for all $y \in N$. Then the problem (2) is well-posed if and only if f is proper. In this case, J_y has a (unique) minimizing point which, for each $y \in N$, is an interior point.

Corollary 2.8
Let M be a connected, open subset of $\mathbb{R}^n$, and let $\varphi : M \to \mathbb{R}$ be a C^1 function, which is proper and bounded from below. Then, if the derivative $\varphi' : M \to (\mathbb{R}^n)^*$ is locally injective and proper, then φ has a unique minimum on M, and φ' is a homeomorphism between M and $(\mathbb{R}^n)^*$.

Remark 2.9
This formulation and conclusion remain valid, *mutatis mutandis*, whenever M is a parallelizable n-manifold.

Corollary 2.10
Let M be an open convex subset of $\mathbb{R}^n$, and let $\varphi : M \to \mathbb{R}$ be a C^1 convex function. Consider the problem to minimize

$$\psi(x) = \varphi(x) - \langle b, x \rangle$$

If the derivative $\varphi' : M \to (\mathbb{R}^n)^*$ is locally injective and proper, then, for each $b \in (\mathbb{R}^n)^*$, ψ achieves a unique minimum on M and ψ' is a homeomorphism between M and $(\mathbb{R}^n)^*$.

3. THE ABSENCE OF SHOCK WAVES IN BURGERS' EQUATION

Consider the (vector) inviscid Burgers' equation

$$\frac{\partial \Pi}{\partial t} = \frac{\partial \Pi}{\partial x}\Pi, \quad \Pi(x, 0) = \Pi(x) \tag{13}$$

where $x \in \mathbb{R}^n$ and Π is a smooth map $\Pi : \mathbb{R}^n \to \mathbb{R}^n$.

We are interested in long-time existence of solutions on arbitrary intervals $[t_0, T]$, with the obstructions to existence being finite escape time or the existence of a shock wave at some time t. We shall first define a variational problem for which, when viewed as an optimal control problem, the Riccati partial differential equation [15] for the gradient of the value function turns out to be the vector inviscid Burgers' equation in backwards time.

For $Q : \mathbb{R}^n \to \mathbb{R}$ a smooth map, the canonical equations

$$\begin{aligned} \dot{x} &= p, \quad x(t_0) = x_0 \in \mathbb{R}^n \\ \dot{p} &= 0, \quad p(T) = -\nabla Q(x(T)) \in \mathbb{R}^n \end{aligned} \tag{14}$$

for the variational problem

$$V(x, t_0) = \inf\left\{ \int_{t_0}^{T} \frac{1}{2} \|\dot{x}\|^2 \, \mathrm{d}t + Q(x(T)) \right\} \tag{15}$$

generate characteristic curves

$$p = \Theta(x(t)) = \Theta(x, t)$$

which satisfy a related equation

$$0 = \dot{p} = \frac{\partial \Theta}{\partial t} + \frac{\partial \Theta}{\partial x}\Theta$$

with terminal condition

$$\Theta(x, T) = -\nabla Q(x)$$

In particular, a solution of Burgers' equation with

$$\Pi(x, 0) = \nabla Q(x)$$

yields a solution (backwards in time)

$$p(T - t) = \Theta(x, T - t) = -\Pi(x, t)$$

of the canonical equations (14). Denote the 'time t' map of the canonical equations by

$$\Phi_t : (x_0, p_0) \mapsto \Phi_t(x, p) = (x(t; x_0), p(t; p_0))$$

In order to define the other ingredient in a well-connected pair, we also consider the two projection maps

$$\text{proj}_1(x, p) = x \quad \text{and} \quad \text{proj}_2(x, p) = p$$

defined on $\mathbb{R}^{2n}$. If Q is C^{k+1}, then

$$M_t = \Phi_t(\text{graph}(\Pi(x, 0)))$$

is a smooth (C^k) n-manifold for all $t \geqslant 0$. Now define the smooth map $f_t : \mathbb{R}^n \to \mathbb{R}^n$ *via*

$$f_t(x_0) = \text{proj}_1\{\Phi_t(x_0, \nabla Q(x_0))\}$$

To say that the problem of solving

$$f_t(x) = y$$

is well-posed is to say that M_t is the graph of a C^k function. If this is the case for all $t \in [t_0, T]$, then (see [26, 27])

$$M_t = \text{graph}(\Pi(x, t)) = \text{graph}(V'(x, t))$$

so that the 'value' function V of the variational problem is C^{k+1} and is a classical solution of the Hamilton–Jacobi Theorem. In general, if V is C^2, then it is C^{k+1}, and, in fact, a shock wave for Burgers' equation occurs precisely when V fails to be twice differentiable, and in this case the (Lagrangian) submanifold M_t is a 'generalized solution' of Burgers' equation.

We now show that these problems are well-connected. Integrating the canonical equations (or the Euler–Lagrange equations) to obtain

$$x(t; x_0) = pt + x_0 \quad \text{for } p \text{ a constant vector in } \mathbb{R}^n$$

one can see that f_t is proper provided either ∇Q is bounded or $\langle \nabla Q(x), x \rangle \geqslant 0$ for all $x \in \mathbb{R}^n$. Now, the critical points of

$$W(x_0, p) = \frac{1}{2}\int_{t_0}^{T} \|\dot{x}\|^2 \, \mathrm{d}t + Q(p(T - t_0) + x_0)$$

are characterized by the terminal constraint equations

$$p = -\nabla Q(p(T - t_0) + x_0)$$

Therefore, every solution x_0 of

$$f_t(x) = y_0$$

determines, in a 1–1 fashion, a critical point

$$p_0 = \text{proj}_2(\Phi_t(x_0, \nabla Q(x_0)))$$

of W, but not necessarily a minimum. Indeed, if x_i satisfy $f_t(x_i) = y_0$ for $i = 1, 2, \ldots, N$, and p_i are the corresponding 'costates', then $(y_0, p_i) \in M_t$ generate extremal trajectories for the variational problems, which may correspond to local minima, local maxima, or inflections, as we shall illustrate by example.

We next investigate when the variational problem has a minimum as its only critical point. Analytically, the Hessian of $W(x_0, p)$ at a critical point is given by

$$D^2 W(x_0, p) = I + (T - t_0) D^2 Q(p(T - t_0) + x_0)$$

In particular, if $D^2 Q(x) > 0$ for $x \in \mathbb{R}^n$, then $W(x_0, p)$ has only non-degenerate minima as critical points.

One can also see this directly from the variational problem. Indeed, suppose that Q is strictly convex. Then, the variational problem (15) is a strictly convex optimization problem defined on an affine subset of a Hilbert space and therefore has a unique minimum point. Moreover, the problem of solving

$$f_t(x) = y$$

is well-posed so that there do not exist shock waves. Conversely, suppose $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is an initial condition for (13), and consider the submanifold

$$\text{graph}(\varphi) = \{(x, p) | p = \varphi(x)\} \subset \mathbb{R}^{2n}$$

where we now consider $\mathbb{R}^{2n}$ as a symplectic manifold with the standard symplectic form $\mathrm{d}x \wedge \mathrm{d}p$. To say that graph($\varphi$) is a Lagrangian submanifold is of course to say that there exists a Q such that

$$\nabla Q(x) = \varphi(x)$$

which also occurs if, and only if, the 1-form

$$\omega = \sum_{i=1}^{n} \varphi_i \, \mathrm{d}x_i$$

is exact, $\omega = \mathrm{d}\varphi$. We call φ Lagrangian in this case. Suppose this is the case and that φ is strictly monotone; i.e.

$$\langle \varphi(x) - \varphi(y), x - y \rangle > 0$$

for $x, y \in \mathbb{R}^n$ with $x \neq y$. In this case, Q is strictly convex. Therefore, there exists no shock waves as we solve (13) with $\Pi(x, 0) = \varphi(x)$ whenever φ is a strictly monotone, Lagrangian map.

If $n = 1$, every function is Lagrangian, so this argument yields a variational proof of the classical fact that shock waves do not occur for monotone increasing initial data.

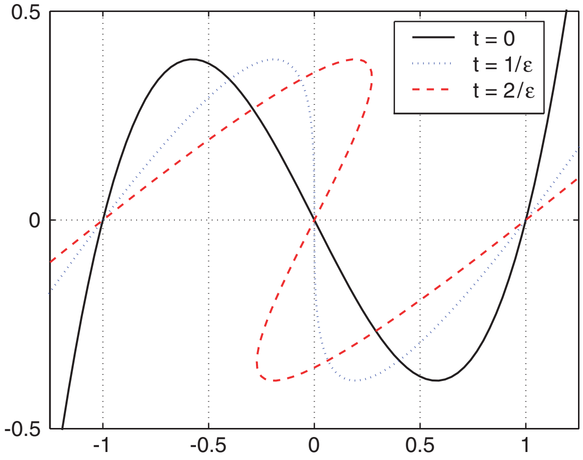

Figure 1. The onset of a shock wave for nonconvex initial data.

Finally, consider the explicit example with

$$Q(x) = \frac{1}{4}(x^4 - 2\varepsilon x^2)$$

When $\varepsilon = 0$, there are no shocks, but for $\varepsilon > 0$, a shock occurs at $T = 1/\varepsilon$. For $T > 1/\varepsilon$ the pitchfork bifurcation for $Q'(x) = 0$ produces a shock wave yielding three extremals, two minima and a maximum, for sufficiently small x, as can be seen in Figure 1.

4. UNIQUENESS IMPLIES EXISTENCE FOR ALGEBRAIC RICCATI EQUATIONS

Global inverse function theorems give criteria for problems to be well-posed. The following example illustrates the use of uniqueness implying existence for the algebraic Riccati equation, replacing a standard argument requiring an analysis of the nonlinear matrix Riccati differential equation.

As is well-known, the algebraic Riccati equation

$$PA + A^{\mathsf{T}}P - PBB^{\mathsf{T}}P = -C^{\mathsf{T}}C \tag{16}$$

plays a crucial role in infinite-horizon linear-quadratic optimal control. Recall that, in that problem, one considers the optimal control problem to infimize

$$\int_0^\infty (\|y(t)\|^2 + \|u(t)\|^2)\,\mathrm{d}t \tag{17}$$

subject to the constraints

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{18}$$

$$y = Cx \tag{19}$$

Here x is square integrable as a function on $(0, \infty)$ taking values in $\mathbb{R}^d$ and u is square integrable as a function on $(0, \infty)$ taking values in $\mathbb{R}^k$. Thus, A, B and C are real matrices of dimensions

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$d \times d$, $k \times k$ and $m \times d$, respectively. Without loss of generality, we may take $m \leqslant d$. We assume that the system (18) is *controllable*; i.e. for any time $t_1 > 0$ and any initial state $x_0 \in \mathbb{R}^d$, there exists a control u defined on the interval $[0, t_1]$, which drives the trajectory of (18) from state x_0 and time $t = 0$ to state 0 at time $t = t_1$. Controllability ensures that the integral (17) has a finite infimum. We also assume that the system is *observable*; i.e. if $y(t) \equiv 0$ on some interval $[0, t_1]$, then $x_0 = 0$.

In the solution of such optimal control problems it is important to show the existence of a unique positive definite solution to (16). Here we shall use Theorem 2.6 to do this. To this end, we first use a standard argument to show uniqueness. In fact, let P be any positive definite solution of (16), and consider the control law

$$u(t) = -B^{\mathsf{T}} P x(t) \tag{20}$$

which yields the solution $x(t) = \mathrm{e}^{\Gamma t} x_0$, where $\Gamma := A - BB^{\mathsf{T}} P$. Then, P is also a positive definite solution to

$$\Gamma^{\mathsf{T}} P + P\Gamma = -PBB^{\mathsf{T}} P - C^{\mathsf{T}} C \tag{21}$$

and consequently, since the right member of the Lyapunov equation (21) is negative semidefinite, Γ must have all its eigenvalues in the closed left half-plane $\operatorname{Re} z \leqslant 0$. However, observability implies that none of these eigenvalues lies on the imaginary axis, for if there were such eigenvalues we could choose the initial value x_0 in the corresponding eigenspace, thus producing a solution $x(t) = \mathrm{e}^{\Gamma t} x_0$ which is periodic; i.e. $x(t_0) = x(t_1)$ for some $t_0 \neq t_1$. Computing the rate of change of the quadratic 'candidate' Lyapunov function $x^{\mathsf{T}} P x$ along this trajectory, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\mathsf{T}} P x = x^{\mathsf{T}} (\Gamma^{\mathsf{T}} P + P\Gamma) x = -x^{\mathsf{T}} (PBB^{\mathsf{T}} P + C^{\mathsf{T}} C) x$$

so integrating from t_0 to t_1 we obtain

$$\int_{t_0}^{t_1} (\|B^{\mathsf{T}} P x\|^2 + \|Cx\|^2) \, \mathrm{d}t = 0$$

However, then $B^{\mathsf{T}} P x \equiv 0$ and $Cx(t) \equiv 0$ on $[t_0, t_1]$, contradicting observability. Consequently, Γ has all its eigenvalues in the open left half-plane, and hence the control law (20) is stabilizing so that, in particular, both x and u are square integrable.

Now, if u is any square-integrable control for which x is also square integrable, then first computing the rate of change of $x^{\mathsf{T}} P x$ along this trajectory, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\mathsf{T}} P x = x^{\mathsf{T}} PAx + u^{\mathsf{T}} PBu + x^{\mathsf{T}} A^{\mathsf{T}} P x + u^{\mathsf{T}} B^{\mathsf{T}} P x$$

Next, integrating from 0 to ∞ yields

$$-x_0^{\mathsf{T}} P x_0 = \int_0^{\infty} (x^{\mathsf{T}} PAx + u^{\mathsf{T}} PBu + x^{\mathsf{T}} A^{\mathsf{T}} P x + u^{\mathsf{T}} B^{\mathsf{T}} P x) \, \mathrm{d}t$$

In particular,

$$\int_0^{\infty} (\|y(t)\|^2 + \|u(t)\|^2) \, \mathrm{d}t = x_0^{\mathsf{T}} P x_0 + \int_0^{\infty} \|u + B^{\mathsf{T}} P x\|^2 \, \mathrm{d}t \tag{22}$$

From this one sees that, if P were to exist, the control law (20) would be optimal for all initial data x_0, resulting in a minimum cost $J(\hat{x}, \hat{u}) = x_0^{\mathsf{T}} P x_0$. Supposing that $\tilde{P}$ is another positive

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definite solution of (16), we obtain, as above, $J(\hat{x},\hat{u}) = x_0^{\mathsf{T}}\tilde{P}x_0$, i.e.

$$x_0^{\mathsf{T}}(\tilde{P} - P)x_0 = 0$$

However, this is valid for all $x_0 \in \mathbb{R}^n$, and hence $\tilde{P} = P$. Therefore, the algebraic Riccati equation (16) has at most one positive definite solution.

To show that the uniqueness of the positive definite solution implies existence of such a solution, we apply Theorem 2.6. To this end, we first assume that $Q := C^{\mathsf{T}}C$ is positive definite. Consider the two n-manifolds, where $n := d(d+1)/2$,

$$M = \{P | P > 0 \text{ and } -PA - A^{\mathsf{T}}P + PBB^{\mathsf{T}}P > 0\}$$

and

$$N = \{Q | Q > 0\}$$

Note that N is connected. We define the continuous map $f : M \to N$ to be

$$f(P) = -PA - A^{\mathsf{T}}P + PBB^{\mathsf{T}}P$$

We have just established that f is injective.

To see that f is proper, we first note that

$$f(\partial M) \subset \partial N$$

Then, all that remains to be proven to establish properness is to show that, if $Q_k \to Q \in N$ as $k \to \infty$ and $P_k \in M$ with $f(P_k) = Q_k$, then $\|P_k\| \leqslant c$ for some constant c. To this end, let $\tilde{u}$ be a control that drives the controllable system (18) from state x_0 at time $t = 0$ to state 0 at time $t = t_1$ and that is identically zero for $t \geqslant t_1$, and let $\tilde{x}$ be the corresponding state trajectory. Then, since P_k is a solution to the algebraic Riccati equation $f(P) = Q_k$, it follows from (22) that

$$x_0^{\mathsf{T}}P_k x_0 \leqslant \int_0^{t_1} (\tilde{x}^{\mathsf{T}}Q_k\tilde{x} + \|\tilde{u}\|^2)\,\mathrm{d}t$$

However, since $Q_k \to Q$, there is a matrix $\tilde{Q}$ such that $Q_k \leqslant \tilde{Q}$. Consequently, for each $x_0 \in \mathbb{R}^n$, there is a bound $c(x_0)$ such that $x_0^{\mathsf{T}}P_k x_0 \leqslant c(x_0)$ for all k. Bounding $x_0^{\mathsf{T}}P_k x_0$ in this way for each x_0 in a basis in $\mathbb{R}^d$ provides an upper bound for P_k and thus establishes the required bound on $\|P_k\|$. Consequently, f is proper, so, by Theorem 2.6, f is a homeomorphism. We also observe that solvability of the equation $f(P) = Q$ and the linear-quadratic optimization problem are well-connected.

In particular, the equation $f(P) = Q$ has a solution that depends continuously on Q. To prove that there exists a solution to the algebraic Riccati equation (16), we need to extend this result to the boundary. Let $\tilde{C}$ be the $d \times d$ matrix obtained from C by amending zero rows as needed, and define the square matrix $C_\varepsilon := \tilde{C} + \varepsilon I$, where $\varepsilon > 0$. If we exchange C for C_ε, we do not change the dynamical system (18), and, since C_ε is full rank, we still have observability. Also, $Q_\varepsilon := C_\varepsilon^{\mathsf{T}}C_\varepsilon \in N$. Next, let P_ε be the unique solution to $f(P) = Q_\varepsilon$. Again applying the control $\tilde{u}$, driving the trajectory of (18) to zero at time $t = t_1$, (22) implies that there is a bound $\mu(x_0)$ for all $x_0 \in \mathbb{R}^n$ such that

$$x_0^{\mathsf{T}}P_\varepsilon x_0 \leqslant \int_0^{t_1} (\|C\tilde{x}(t)\|^2 + \|\tilde{u}(t)\|^2)\,\mathrm{d}t + \varepsilon\int_0^{t_1} \tilde{x}^{\mathsf{T}}(\tilde{C}^{\mathsf{T}} + \tilde{C} + \varepsilon I)\tilde{x}\,\mathrm{d}t \leqslant \mu(x_0)$$

for all $\varepsilon \in [0,\delta]$. Precisely as above, this provides an upper bound for P_ε on $[0,\delta]$. Hence, as $\varepsilon \to 0$, some subsequence of P_ε tends to a limit $P_0 \geqslant 0$, which must clearly satisfy (16). Since

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$\Gamma_\varepsilon := A - BB^{\mathsf{T}}P_\varepsilon$ has all its eigenvalues in the open right half-plane, the limit $\Gamma_0 := A - BB^{\mathsf{T}}P_0$ has all its eigenvalues in the closed left half-plane. However, just as above, we can show that observability ensures that none of these eigenvalues can lie on the imaginary axis. Consequently, (22) holds for $P = P_0$ and $u = -B^{\mathsf{T}}P_0 x$ so that

$$x_0^{\mathsf{T}} P_0 x_0 = \int_0^\infty (\|y(t)\|^2 + \|u(t)\|^2)\,\mathrm{d}t$$

Therefore, if there were an $x_0 \neq 0$ such that $x_0 P_0 x_0 = 0$, this would contradict observability. Hence, P_0 must be positive definite.

Consequently, there is always a unique positive definite solution P of the Riccati equation (16), which depends continuously on C.

5. INTERIOR-POINT METHODS FOR LINEAR PROGRAMMING PROBLEMS

Recall that linear programming problems are often given in the standard form

$$\min c^{\mathsf{T}}x \quad \text{subject to } Ax = b, \quad x \geqslant 0 \tag{23}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ are given and $x \in \mathbb{R}^n$ is the variable. For simplicity, we assume that the matrix A is fixed and has full rank $m \leqslant n$. The corresponding dual problem is

$$\max b^{\mathsf{T}}u \quad \text{subject to } A^{\mathsf{T}}u \leqslant c$$

where $u \in \mathbb{R}^m$ is the variable. This can also be written

$$\max b^{\mathsf{T}}u \quad \text{subject to } A^{\mathsf{T}}u + s = c, \quad s \geqslant 0 \tag{24}$$

by introducing slack variables $s_1, s_2, \ldots, s_n$. From now on, we shall refer to (23) as the *primal* problem and (24) as the *dual* problem.

It is well-known that the primal problem has an optimal solution if and only if the dual one does, and this happens if and only if the the *primal-dual feasibility set*

$$\mathscr{F} = \{(x, u, s) | Ax = b, A^{\mathsf{T}}u + s = c, x \geqslant 0, s \geqslant 0\}$$

is non-empty. Then $(\hat{x}, \hat{u}, \hat{s}) \in \mathscr{F}$ is optimal for the dual and the primal problems if and only if the *complementary-slackness condition*

$$\hat{x}_k \hat{s}_k = 0, \quad k = 1, 2, \ldots, n \tag{25}$$

is fulfilled. Let $M = \{(x, u, s) \in \mathbb{R}^{2n+m} | x > 0, s > 0\}$.

Introducing the notation $a = \left[\begin{smallmatrix} b \\ c \end{smallmatrix}\right]$, we also consider the strictly feasible set

$$M_a = \{(x, u, s) \in \mathscr{F} | x > 0, s > 0\}$$

Then $M_a = g^{-1}(a)$ where $g : M \to L := \mathbb{R}^{n+m}$ is defined *via*

$$(x, u, s) \mapsto \begin{bmatrix} Ax \\ A^{\mathsf{T}}u + s \end{bmatrix}$$

On M_a the variable u is uniquely determined by s, and therefore we parameterize a point in M_a by the co-ordinate pair (x, s). A point in M_a is called an *interior point*. The tangent space to M_a

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at an interior point (x, s) is given by

$$T_{(x,s)}(M_a) = \{(h, k) \in \mathbb{R}^n \times \mathbb{R}^n | h \in \ker A, k \in (\ker A)^\perp\}$$

In fact, since $\operatorname{Im} A^\mathsf{T} = (\ker A)^\perp$, $x_1 - x_2 \in \ker A$ and $s_1 - s_2 \in (\ker A)^\perp$ for all (x_1, s_1) and (x_2, s_2) in M_a. In particular,

$$(x_1 - x_2)^\mathsf{T}(s_1 - s_2) = 0 \tag{26}$$

Since $\dim T_{(x,s)}(M_a) = n$, M_a is an n-manifold.

The basic idea in *interior-point methods* is to construct a parameterized set of interior points satisfying

$$x_k s_k = \tau_k, \quad k = 1, 2, \ldots, n \tag{27}$$

where the parameters $\tau_1, \tau_2, \ldots, \tau_n$ are positive real numbers. This set is called the *central path*, and the idea is to construct a sequence of points in this set converging to an optimal solution as $\tau := (\tau_1, \tau_2, \ldots, \tau_n)^\mathsf{T} \to 0$. For details, see, e.g. [28], where, however, τ is chosen so that $\tau_1 = \tau_2 = \cdots = \tau_n$. Here, we prefer the more general parameterization.

Now, define $\mathbb{R}^n_+$ to be the n-manifold of vectors $\tau := (\tau_1, \tau_2, \ldots, \tau_n)^\mathsf{T}$ such that $\tau_k > 0$, $k = 1, 2, \ldots, n$. Next, consider the smooth function $f : M_a \to N := \mathbb{R}^n_+$ given by

$$f(x, s) = \begin{bmatrix} x_1 s_1 \\ x_2 s_2 \\ \vdots \\ x_n s_n \end{bmatrix}$$

If M_a is non-empty, f is proper. To see this, note that $f^{-1}(K)$ is closed by continuity, so it just remains to prove that $f^{-1}(K)$ is bounded. Let $(\tilde{x}, \tilde{s}) \in M_a$, and let $(x, s) \in f^{-1}(K)$ be arbitrary. In view of (26), $(x - \tilde{x})^\mathsf{T}(s - \tilde{s}) = 0$, and hence

$$x^\mathsf{T} s + \tilde{x}^\mathsf{T} \tilde{s} = \tilde{x}^\mathsf{T} s + \tilde{s}^\mathsf{T} x \geqslant \varepsilon \mathrm{e}^\mathsf{T}(x + s)$$

where $e := (1, 1, \ldots, 1)^\mathsf{T}$ and where

$$\varepsilon = \min\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n\} > 0$$

Consequently,

$$0 < \mathrm{e}^\mathsf{T}[x + s] \leqslant \frac{1}{\varepsilon} \mathrm{e}^\mathsf{T}[f(x, s) + f(\tilde{x}, \tilde{s})]$$

which is bounded since K is. Hence, $f^{-1}(K)$ is bounded, as claimed.

We shall now construct a variational problem which is well-connected to f. To this end, for each $\tau \in \mathbb{R}^n_+$, let $J_\tau : M_a \to \mathbb{R} \cup \{\infty\}$ be the function

$$J_\tau(x, s) = x^\mathsf{T} s - \sum_{k=1}^{n} \tau_k \log(x_k s_k) \tag{28}$$

and consider the problem of solving the problem

$$\inf_{M_a} J_\tau(x, s) \tag{29}$$

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The function J_τ is strictly convex for each $\tau \in \mathbb{R}^n_+$. In fact, since the functions $-\log x_k s_k = -\log x_k - \log s_k$ are strictly convex, convexity of J_τ follows from the fact that the function $g(x,s) = x^{\mathsf{T}} s$ is linear on M_a. To see this, note that, by (26),

$$[x_2 + \lambda(x_1 - x_2)]^{\mathsf{T}}[s_2 + \lambda(s_1 - s_2)] = x_2^{\mathsf{T}} s_2 + \lambda x_2^{\mathsf{T}}(s_1 - s_2) + \lambda(x_1 - x_2)^{\mathsf{T}} s_2$$

for $(x_i, s_i) \in M_a$, $i = 1, 2$, which is the same as

$$g(\lambda x_1 + (1-\lambda)x_2, \lambda s_1 + (1-\lambda)s_2) = \lambda g(x_1, s_1) + (1-\lambda) g(x_2, s_2)$$

In fact, in view of (26), $x_2^{\mathsf{T}}(s_1 - s_2) = x_1^{\mathsf{T}}(s_1 - s_2)$.

Next, note that

$$\frac{\partial J_\tau}{\partial x_k} = s_k - \frac{\tau_k}{x_k}, \quad k = 1, 2, \ldots, n$$

$$\frac{\partial J_\tau}{\partial s_k} = x_k - \frac{\tau_k}{s_k}, \quad k = 1, 2, \ldots, n$$

and hence the directional derivative of J_τ in $(x,s) \in M_a$ in the direction (h,k) is given by

$$dJ_\tau(x, s; h, k) = \sum_{k=1}^{n} (x_k s_k - \tau_k)\left[\frac{h_k}{x_k} + \frac{k_k}{s_k}\right]$$

Consequently, $(\hat{x}, \hat{s}) \in M_a$ is a critical point for the optimization problem (29) whenever $(\hat{x}, \hat{s})$ is a solution of

$$f(x,s) = \tau \tag{30}$$

so that (30) is a branch of $dJ_\tau = 0$. In particular, the family of optimization problems (29) and the family of equations (30) form a well-connected pair. Since J_τ is strictly convex, it has at most one critical point, which is a minimum.

By Theorem 2.2, J_τ has an interior minimizing point, and the map f is a homeomorphism between M_a and $\mathbb{R}^n_+$. In particular, as $\|\tau\| \to 0$, (x_τ, s_τ) tends to a limit $(\hat{x}, \hat{s}) \in \overline{M_a}$, which satisfies the complementary-slackness condition (25) and hence is optimal for the primal and dual linear programming problems (23) and (24).

Remark 5.1

We know that f is a homeomorphism so that the problem of solving (30) is well-posed. We also know that the unique solution of (30) is the unique minimizing interior point for the strictly convex functional (28). In fact, one can actually prove smooth dependence of solutions on the data in this particular case. Identifying $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$ using the standard inner product, consider the subsets of $M_a \times N$ defined by the following two conditions:

$$dJ_\tau(x,s) = 0 \tag{31}$$

$$\tau - f(x,s) = 0 \tag{32}$$

The second set, being the graph of a smooth function is a smooth submanifold of dimension n. Since $(\partial/\partial\tau) dJ_\tau(x,s)$ is full rank, the first set is also a submanifold of dimension n, which is everywhere locally the graph of a smooth function of (x,s). One such branch is given by (32), and therefore graph(f) is a connected component of the submanifold (31). Now, consider the projection $p_2(x,s,\tau) = \tau$ restricted to this connected component of (31). The Jacobian of this map, computed with respect to (x, s) at the point (x, s, τ) is $D^2 J_\tau(x, s)$, which is positive definite by strict convexity of J_τ. Therefore, the Jacobian Jac(f) is everywhere non-singular on M_a. Since f is proper, f is a diffeomorphism by Hadamard's Theorem.

Theorem 5.2

For the interior-point method for linear programming, the problem of solving $f(x^i, s^i) = \tau^i$ on the (generalized) central path $\tau = (\tau_1, \tau_2, \ldots, \tau_n)^\mathsf{T} \to 0$ as $i \to \infty$ is well-posed. In particular, the solution sequence (x^i, s^i) exists, is unique, and lies in a compact set. Moreover, the solution (x, s) to (30) is a smooth function of τ. Indeed, (x, s) is the unique minimizing interior point for the strictly convex functional

$$J_\tau(x, s) = x^\mathsf{T} s - \sum_{k=1}^{n} \tau_k \log(x_k s_k)$$

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